Marshallian, hicksian, expenditure and indirect utility functions for quasilinear preferences

Consider the case of a consumer with preferences over two goods x and y, represented by the following utility function $U(x, y) = \ln(x) + y$. The price of good x is \$10 and that of good y is \$4. The consumer's income is \$100. The following is requested:

1. Find the Marshallian demand function for each good. Interpret them economically.
2. Find the indirect utility function and from it the minimum expenditure function. Economically interpret both functions.
3. Find the Hicksian demand functions from the minimum expenditure function. Economically interpret both functions.
4. Suppose the price of good x increases to \$15. Analyze the changes in demand according to the Marshallian and Hicksian demands. What is the reason for this? Justify your answer.

Solution

1. Set up the budget constraint:

$$M = P_x x + P_y y$$

Set up the Lagrangian to maximize utility:

$$L = \ln(x) + y - \lambda(P_x x + P_y y - M)$$

$$\frac{\partial L}{\partial x} = \frac{1}{x} - P_x\lambda = 0 \implies \lambda = \frac{1}{xP_x}$$

$$\frac{\partial L}{\partial y} = 1 - P_y\lambda = 0 \implies \lambda = \frac{1}{P_y}$$

$$\frac{\partial L}{\partial \lambda} = P_x x + P_y y - M = 0$$

Equalize λ from the first two equations:

$$\frac{1}{xP_x} = \frac{1}{P_y}$$

$$x = \frac{P_y}{P_x}$$

Insert into the budget constraint:

$$P_x \frac{P_y}{P_x} + P_y y - M = 0$$

$$P_y + P_y y - M = 0$$

$$y = \frac{M - P_y}{P_y}$$

With this, I construct the optimal demands:

$$y^m = \begin{cases} \frac{M-P_y}{P_y} & \text{if } M \geq P_y \\ 0 & \text{if } M < P_y \end{cases}$$

$$x^m = \begin{cases} \frac{P_y}{P_x} & \text{if } M \geq P_y \\ \\ \frac{M}{P_x} & \text{if } M < P_y \end{cases}$$

The demands are divided since if the price of P_y is too high, there cannot be negative demand for y, therefore the demand would be 0. In this case, replacing with the corresponding values, we find ourselves in the first segment of the divided demand:

$$x^m = 0.4$$

$$y^m = 24$$

The optimal basket given this individual's budget, preferences, and prices is $(0.4, 24)$. This basket ensures that the individual maximizes utility given their budget.

2. The indirect utility function is obtained by replacing the Marshallian demands in the utility function:

$$v(P_x, P_y, M) = \ln(x^m) + y^m = \begin{cases} \ln(\frac{P_y}{P_x}) + \frac{M-P_y}{P_y} & \text{if } M \geq P_y \\ \ln(\frac{M}{P_x}) & \text{if } M < P_y \end{cases}$$

With the data from this exercise:

$$v(P_x, P_y, M) = 23.08$$

The indirect utility function represents the maximum level of utility that a consumer can achieve given a set of prices and a specified income level. Mathematically, this function reflects the maximum utility that a consumer obtains by solving their utility maximization problem subject to their budget constraint. The indirect utility function depends on the prices of the goods and the consumer's income, but not directly on the quantities of goods consumed.

Formally, if $U(x, y)$ is the utility function that depends on the quantities of goods x and y, and P_x and P_y are the prices of these goods with an income M, the indirect utility function $v(P_x, P_y, M)$ is defined as:

$$v(P_x, P_y, M) = \max_{x,y}\{U(x, y) \mid P_x x + P_y y \leq M\}$$

This function is useful for analyzing how the maximum achievable utility changes when prices and income vary, keeping consumer preferences fixed.

To obtain the minimum expenditure function, we have to invert the indirect utility function, assuming we are in the first segment:

$$e(P_x, P_y, u) = uP_y + P_y - P_y \ln(\frac{P_y}{P_x})$$

Assuming we are in the second segment:

$$e(P_x, P_y, u) = e^u P_x$$

Then:

$$e(P_x, P_y, u) = \ln(x^m) + y^m = \begin{cases} uP_y + P_y - P_y \ln(\frac{P_y}{P_x}) & \text{if } M \geq P_y \\ e^u P_x & \text{if } M < P_y \end{cases}$$

With the prices and income from this exercise:

$$e(P_x, P_y, u) = 4u - 7.67$$

The minimum expenditure function, $e(P_x, P_y, u)$, represents the minimum expenditure necessary to reach a utility level u given the prices P_x and P_y. Mathematically, it is defined as:

$$e(P_x, P_y, u) = \min_{x,y}\{P_x x + P_y y \mid U(x, y) \geq u\}$$

3. To obtain the Hicksian demand functions we derive the minimum expenditure function. Assuming we are in the first segment:

$$\frac{\partial e(P_x, P_y, u)}{\partial P_x} = -P_y \frac{P_x}{P_y} \left(\frac{P_y}{P_x^2}\right) = \frac{P_y}{P_x} = -h_x$$

$$\frac{\partial e(P_x, P_y, u)}{\partial P_y} = u + 1 - (1 \ln(P_y/P_x) + P_y \frac{P_x}{P_x P_y}) = u - \ln(P_y/P_x) = h_y$$

Assuming we are in the second segment:

$$\frac{\partial e(P_x, P_y, u)}{\partial P_x} = e^u = h_x$$

$$\frac{\partial e(P_x, P_y, u)}{\partial P_y} = 0 = h_y$$

$$h_x(p_x, p_y, u) = \begin{cases} \frac{P_y}{P_x} & \text{if } M \geq P_y \\ \\ e^u & \text{if } M < P_y \end{cases}$$

$$h_y(p_x, p_y, u) = \begin{cases} u - \ln(P_y/P_x) & \text{if } M \geq P_y \\ \\ 0 & \text{if } M < P_y \end{cases}$$

With the data from this exercise:

$$h_x = 4/10 = 0.4$$
$$h_y = 23.28 + 0.92 = 24.20$$

The Hicksian demands represent the amount of each good that the consumer would wish to purchase to reach a specific utility level at the lowest possible cost, given price constraints. Unlike Marshallian demand, Hicksian demands are "compensated" in that they adjust to maintain the utility level constant despite price changes, thus removing the income effect and retaining only the substitution effect.

4. If the price of good x now is 11, then the Marshallian demands will be as follows:

$$x^m = \frac{P_y}{P_x} = \frac{4}{11}$$

$$y^m = \frac{M - P_y}{P_y} = \frac{100 - 4}{4} = 24$$

This provides a utility of:

$$v(P_x, P_y, M) = 22.99$$

And the Hicksian:

$$h_x = 4/11$$

$$h_y = 22.99 + 1.01 = 24$$

The result indicates that the demand for good x is reduced in both the Hicksian and Marshallian demands. Additionally, utility is reduced. What is happening here is that the individual consumes less of good x because it is now relatively more expensive compared to good y. There is no income effect in the Marshallian demand for good x since this demand does not depend on the available income. Therefore, both the Hicksian and Marshallian demands are reflecting the same change, namely a reduction in the demand for x because it is relatively more expensive. This result is common in quasilinear utility functions, for the good that enters quasilinearly there is no income effect.

Substitution Effect: Good x becomes more expensive compared to other goods, leading the consumer to substitute x for cheaper goods. This reduces the demanded quantity of x.

Income Effect: The increase in the price of x effectively reduces the real income of the consumer (since they can now buy less with the same money), which may lead to an additional reduction in the demand for x if it is a normal good.

If we calculate the substitution and income effects:

$$\text{Substitution effect } = x(P_x', M') - x(P_x, M)$$
$$\text{Income effect} = x(P_y', M) - x(P_x', M')$$

Substitution effect: Measures the change in the demanded quantity of the good when its price changes, keeping the utility level constant (as if income were adjusted to keep utility constant). Here, $x(P_x', M')$ represents the demanded quantity of good x at the new price P_x' and with an adjusted income M' that maintains utility constant, while $x(P_x, M)$ represents the demanded quantity of good x at the original price P_x with the original income M.

Income effect: Measures how the demanded quantity of the good changes due to a change in income, after considering the price change (i.e., after adjusting for the substitution effect). $x(P_y', M)$ appears to have a typographical error in your expression, as it normally should refer to the demanded quantity with the new price and the original income, like $x(P_x', M)$, indicating the demand under the new price but without income adjustment. Then, $x(P_x', M')$ represents the demand with the new price and new income. Note that as the demand for x^m does not depend on income, there is no need to find M'

We calculate:

$$x^m(P_x', M') = \frac{4}{11}$$
$$x^m(P_x', M) = \frac{4}{11}$$

Thus:

$$\text{Substitution effect } = \frac{4}{11} - 0.4 = -2/55$$
$$\text{Income effect} = \frac{4}{11} - \frac{4}{11} = 0$$

In conclusion, we can also see this with the Hicksian demand:

$$h_x' - h_x = \frac{4}{11} - \frac{4}{10} = -2/55$$

$$x'^m - x^m = \frac{4}{11} - \frac{4}{10} = -2/55$$

The reduction in the demand for x^m is due only to the substitution effect and is equal to $-2/55$.